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J. Differential Equations 246 (2009) 2415–2433

Journal of Differential Equations

www.elsevier.com/locate/jde

Criteria to bound the number of critical periods

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Received 9 May 2008; revised 27 June 2008

Available online 24 July 2008

Abstract

In the present paper we study the period function of centers of potential systems. We obtain criteria to bound the number of critical periods. In case that the system is polynomial, our result enables to tackle the problem from a purely algebraic point of view, since it allows to bound the number of critical periods by counting the zeros of a polynomial. To illustrate its applicability some new and old results are proved.

MSC: 34C07; 34C08; 34C23

Keywords: Center; Period function; Critical period; Chebyshev system

1. Introduction and statement of the main result

In this paper we study the period function of centers and we are interested in the case in which it is not monotone, i.e., the center has critical periods. To our knowledge, the key point in almost all the results appearing in the literature dealing with a family of centers with critical periods is that the period function satisfies some kind of Picard–Fuchs differential equation. Let us quote for instance the works of Yulin Zhao for two different families of quadratic centers [24,25] or the papers of Chow and Sanders [4] and Gavrilov [10] on the family of cubic potential centers.

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1 The author is partially supported by the MEC/FEDER grants MTM2005-02139 and MTM2005-06098 and the CIRIT grant 2005SGR-00550.
2 The author is partially supported by the MEC/FEDER grant MTM2005-06098-C02-01 and the CIRIT grant 2005SGR-00550.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2008.07.002

This approach is very complicated and there is little hope of success in its application to study other families of centers. The present paper is devoted to prove a criterion that allows to bound the number of critical periods of the centers (degenerate or not) of potential systems.

More concretely we consider Hamiltonian differential systems of the form

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = V'(x), \end{cases} \tag{1}$$

where V is an analytic function in a neighbourhood of $x = 0$ with $V(x) = \frac{1}{2m}x^{2m} + \mathrm{o}(x^{2m})$. The singular point at the origin of this *potential system* is a *center* and its *period annulus*, that we denote by $\mathcal{P}$, is the biggest punctured neighbourhood of the origin foliated by periodic orbits. The *period function* assigns to each periodic orbit inside $\mathcal{P}$ its period. Note that the solutions curves of system (1) are inside the energy levels of the Hamiltonian, $\{\frac{1}{2}y^2 + V(x) = h\}$. The period annulus has two boundary components; the center itself, with energy level $h = 0$, and a polycycle with energy level $h = h_0 > 0$. (Note that $h_0 = +\infty$ in case of an unbounded period annulus.) For each $h \in (0, h_0)$ we denote by γ_h the periodic orbit of $\mathcal{P}$ inside the energy level h. In order to study the period function we take the parametrization of the set of periodic orbits given by the energy, i.e., $h \mapsto T(h) :=$ period of γ_h. Accordingly,

$$T(h) = \int_{\gamma_h} \frac{dx}{y} \quad \text{for each } h \in (0, h_0).$$

The *critical periods* are the zeros of the derivative of T. Here we use the parametrization of $\mathcal{P}$ given by the energy level of the Hamiltonian, but it can be shown that the number, character (maximum or minimum) and distribution of the critical periods do not depend on the particular parametrization used. In what follows we will denote the projection of $\mathcal{P}$ on the x-axis by $\mathcal{I}$. Clearly $\mathcal{I} = (x_\ell, x_r)$ with $x_\ell < 0 < x_r$.

Since $xV'(x) > 0$ for all $x \in \mathcal{I} \setminus \{0\}$, there exists an analytic involution σ on $\mathcal{I}$ such that

$$V(x) = V\big(\sigma(x)\big) \quad \text{for all } x \in \mathcal{I}.$$

(Recall that a mapping σ is said to be an *involution* if $\sigma \circ \sigma = \mathrm{Id}$ and $\sigma \neq \mathrm{Id}$.) If f is an analytic function on $\mathcal{I}$, then we define the σ*-even part* and σ*-odd part* of f by

$$\mathscr{P}_\sigma(f)(x) = \frac{f(x) - f(\sigma(x))\sigma'(x)}{2} \quad \text{and} \quad \mathscr{I}_\sigma(f)(x) = \frac{f(x) + f(\sigma(x))\sigma'(x)}{2},$$

respectively. We say that f is σ*-even* if $\mathscr{P}_\sigma(f) = f$ and that f is σ*-odd* if $\mathscr{I}_\sigma(f) = f$. Now with this notation we can state the main result of the present paper:

Theorem A. *Let us consider the period function $T(h)$ of the center of system* (1) *and, taking $\mu_1 = -\frac{1}{2} + (\frac{V}{V'})'$, define $\mu_{k+1} = \frac{1}{2}\mu_k + \frac{1}{2k-1}(\frac{\mu_k V}{V'})'$. Then, for any $k \geqslant 1$, the following equalities hold*:

$$T'(h) = \frac{1}{h^k} \int_{\gamma_h} \mu_k(x) y^{2k-3}\, dx = \frac{1}{h^k} \int_{\gamma_h} \mathscr{P}_\sigma(\mu_k)(x) y^{2k-3}\, dx.$$

Moreover, if for some $k \geqslant 1$ the number of zeros of $\mathscr{P}_\sigma(\mu_k)$ on $(0, x_r)$, counted with multiplicities, is $n < k$, then the number of critical periods (i.e., zeros of T'), counted with multiplicities, is at most n.

For each $k \geqslant 1$, the function μ_k defined recursively in Theorem A provides a criterion to bound the number of critical periods of system (1). We point out however that the corresponding criterion can only be applied as long as the number of zeros of $\mathscr{P}_\sigma(\mu_k)$ on $(0, x_r)$, counted with multiplicities, is smaller than k. This is the reason why, for instance, μ_1 can only be used to detect monotonicity (i.e., absence of critical periods).

In the literature there are many papers dealing with conditions to guarantee that a center of a potential system has no critical periods (see [3,7,17,18,20,22,23] and references therein). However very few papers go beyond monotonicity in the study of the qualitative properties of the period function. To our knowledge, only the papers of Bonorino et al. [1] and Sabatini [19], that provide criteria in order that a center has at most one critical period. (In fact both results give conditions under which T'' does not vanish.)

We are convinced that Theorem A will be useful to go further in the study of the qualitative properties of the period function, and not only in the context of potential systems. For instance, see Lemma 5 in [9], any center with a first integral quadratic in y, i.e., of the form $H(x, y) = A(x) + B(x)y + C(x)y^2$, and having an integrating factor only depending on x can be brought to a potential system. Since any *reversible quadratic center* has this property, Theorem A may contribute to prove Chicone's conjecture [2], which claims that these centers have at most two critical periods (see [14,16,21] for partial results on the issue).

The paper is organized in the following way. Section 2 is devoted to the proof of Theorem A, which strongly relies on a criterion given in [11] that provides a sufficient condition for a collection of Abelian integrals to be a Chebyshev system. Moreover at the end of the section we prove a necessary condition, see Theorem 2.6, in order that a center has no critical periods. In Section 3 we give several applications of Theorem A. Our aim is twofold. Firstly, by studying already known results [4,10,12], to show that our criterion is easy to apply and avoids long computations. Certainly this is more evident when the potential V is an even function because then $\sigma = -\mathrm{Id}$. Our second goal is to show that in fact it is not necessary to know explicitly σ to apply our result and to this end, in the last example, we tackle a case in which $\sigma \neq \mathrm{Id}$. In short, thanks to Theorem A, bounding the number of critical periods turns into a purely algebraic problem, namely, to count zeros of a polynomial (sometimes depending on a parameter). To solve this latter problem we shall frequently use Sturm's Theorem and the notion of resultant and discriminant. For the reader's convenience, we conclude the paper with Appendix A in which we recall these tools.

2. Proof of the main result

The reader is referred to [15] for details about the following definition and related results on the issue.

Definition 2.1. Let $f_0, f_1, \ldots, f_{n-1}$ be analytic functions on an open interval L of $\mathbb{R}$.

(a) $(f_0, f_1, \dots, f_{n-1})$ is a *Chebyshev system* (in short, T-system) on L if any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{n-1} f_{n-1}(x) = 0$$

has at most $n-1$ isolated zeros on L.

(b) $(f_0, f_1, \dots, f_{n-1})$ is a *complete Chebyshev system* (in short, CT-system) on L if $(f_0, f_1, \dots, f_{k-1})$ is a T-system for all $k = 1, 2, \dots, n$.

(c) $(f_0, f_1, \dots, f_{n-1})$ is an *extended complete Chebyshev system* (in short, ECT-system) on L if, for all $k = 1, 2, \dots, n$, any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{k-1} f_{k-1}(x) = 0$$

has at most $k-1$ isolated zeros on L counted with multiplicities.

(Let us mention that, in these abbreviations, "T" stands for Tchebycheff, which in some sources is the transcription of the Russian name Chebyshev.)

The next result refers to the notion of σ-parity that we introduced before the statement of Theorem A.

Proposition 2.2. *Let σ be an analytic involution on an interval $L = (a, b)$ with $a < 0 < b$ and $\sigma(0) = 0$. Let f be an analytic σ-even function on L. Then f has at most n zeros on $(0, b)$ counted with multiplicities if and only if there exist $g_0, g_1, \dots, g_{n-1}$ analytic σ-even functions on L such that $(g_0, g_1, \dots, g_{n-1}, f)$ is an ECT-system on $(0, b)$.*

Proof. The sufficiency follows from the definition of ECT-system. In order to prove the necessity, let us define $\kappa(x) = \frac{x-\sigma(x)}{2}$, which is clearly an analytic diffeomorphism on L because $\sigma'(x) < 0$ for all x. Moreover $\kappa(\sigma(x)) = -\kappa(x)$, so that $\kappa(x)^2 = \kappa(\sigma(x))^2$. Taking this into account, it is easy to show that if f is a σ-even function, then so it is $\frac{f(x)}{\kappa(x)^2-a}$ for any $a \in \mathbb{R}$.

Suppose first that f has exactly n zeros on $(0, b)$ counted with multiplicities, say $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{n-1}$. Define $g_n = f$ and, for $i = 1, 2, \dots, n$, $g_{i-1}(x) = \frac{g_i(x)}{\kappa(x)^2-\kappa(x_{i-1})^2}$, which are σ-even functions by the above observation. On the other hand, since $g_i(x_{i-1}) = g_i(\sigma(x_{i-1})) = 0$ and $\kappa(x)^2 = \kappa(x_{i-1})^2$ if and only if $x = x_{i-1}$ or $x = \sigma(x_{i-1})$, it follows that g_{i-1} is well defined and analytic on L for all $i = 1, 2, \dots, n$. Note in addition that g_0 does not vanish on $(0, b)$ and, for $i = 1, 2, \dots, n$,

$$g_i(x) = g_0(x) \prod_{j=0}^{i-1} \bigl(\kappa(x)^2 - \kappa(x_j)^2\bigr).$$

Accordingly, $(g_0, g_1, \dots, g_n, f)$ is an ECT-system on $(0, b)$ if and only if so it is

$$\left(1, \kappa(x)^2 - \kappa(x_0)^2, \bigl(\kappa(x)^2 - \kappa(x_0)^2\bigr)\bigl(\kappa(x)^2 - \kappa(x_1)^2\bigr), \dots, \prod_{j=0}^{n-1} \bigl(\kappa(x)^2 - \kappa(x_j)^2\bigr)\right).$$

The latter is obvious because the ith entry in the above n-tuple is the composition of an even polynomial of degree exactly $2(i-1)$ with the diffeomorphism κ.

Finally in case that f has $m<n$ zeros on $(0,b)$ counted with multiplicities, we construct $g_0, g_1, \dots, g_{m-1}$ exactly as before. Then, taking any $a_1, a_2, \dots, a_{n-m} \in \mathbb{R}$, we define

$$g_{m+i}(x) = f(x)\prod_{j=1}^{i}\bigl(\kappa(x)^2 - a_j\bigr) \quad \text{for } i = 1,2,\dots,n-m.$$

These $n-m$ functions are σ-even as well and by the same argument as before $(g_0, g_1, \dots, g_m, f, g_{m+1}, \dots, g_n)$ is an ECT-system on $(0,b)$. □

As a matter of fact, chronologically we proved first the result that we state next. However writing the final version of the paper we realized that it is not necessary. We decided to state it anyway for the sake of completeness because it is in some sense not so specific as Proposition 2.2.

Proposition 2.3. *Let f be an analytic function on an interval L. Then f has at most n zeros on L counted with multiplicities if and only if there exist $g_0, g_1, \dots, g_{n-1}$ analytic functions on L such that $(g_0, g_1, \dots, g_{n-1}, f)$ is an ECT-system on L.*

We only sketch the proof of this result because it is easier than the previous one. Indeed, to show the necessity in case that f has exactly n zeros on L counted with multiplicities, say $x_0 \leqslant x_1 \leqslant \dots \leqslant x_{n-1}$, we define $g_n = f$ and $g_{i-1}(x) = \frac{g_i(x)}{x - x_{i-1}}$ for $i = 1,2,\dots,n$. From here on the proof follows the same ideas as the one of Proposition 2.2. The next result is a particular case of Theorem B in [11]. Let us point out however that, for convenience, we adapt its statement to the notation we use in the present paper.

Theorem 2.4. *Let $f_0, f_1, \dots, f_{n-1}$ be analytic functions on $\mathcal{I}$ and define*

$$I_i(h) = \int_{\gamma_h} f_i(x) y^{2k-3}\, dx \quad \textit{for } i = 0,1,\dots,n-1.$$

Then, if $(\mathscr{P}_\sigma(f_0), \mathscr{P}_\sigma(f_1), \dots, \mathscr{P}_\sigma(f_{n-1}))$ is a CT-system on $(0,x_r)$ and $k \geqslant n$, $(I_0, I_1, \dots, I_{n-1})$ is an ECT-system on $(0,h_0)$.

The next result is the last ingredient in the proof of Theorem A.

Lemma 2.5. *If g is any analytic function such that $x \mapsto \frac{g}{V'}(x)$ is regular at $x=0$, then it holds*

$$\int_{\gamma_h} g(x) y^{k-2}\, dx = \frac{1}{k}\int_{\gamma_h}\left(\frac{g}{V'}\right)'(x) y^k\, dx \quad \textit{for any } k \in \mathbb{N}.$$

Proof. Note first that, on account of $\frac{dy}{dx} = -\frac{V(x)}{y}$,

$$d\bigl(\ell(x)y^k\bigr) = \ell'(x)y^k\, dx + k\ell(x)y^{k-1}\, dy = \bigl(\ell'(x)y^k - k\ell(x)V'(x)y^{k-2}\bigr)\, dx.$$

Accordingly

$$\int_{\gamma_h} \ell'(x) y^k \, dx = k \int_{\gamma_h} \ell(x) V'(x) y^{k-2} \, dx$$

for any analytic function ℓ on $\mathcal{I}$. Now the result follows taking $\ell = \frac{g}{V'}$ in the above equality. $\square$

We are now in position to prove the main result of the paper.

Proof of Theorem A. We begin by showing the integral expressions of the derivative of the period function. We will prove the first equality by induction in k. To this end, for each $h \in (0, h_0)$, denote the projection of the periodic orbit γ_h on the x-axis by (x_h^-, x_h^+). Let us define in addition

$$g(x) = \operatorname{sgn}(x) \sqrt[2m]{V(x)} = x \sqrt[2m]{\frac{V(x)}{x^{2m}}},$$

which is an analytic diffeomorphism on $\mathcal{I}$ because $V'(x) \neq 0$ for all $x \in I \setminus \{0\}$ and $V(x) = \frac{1}{2m} x^{2m} + \mathrm{o}(x^{2m})$. Moreover, due to $g(x)^{2m} = V(x)$, it is clear that $g(x_h^{\pm}) = \pm \sqrt[2m]{h}$. Taking $\gamma_h \subset \{\frac{1}{2} y^2 + V(x) = h\}$ into account, we get

$$T(h) = \int_{\gamma_h} \frac{dx}{y} = \sqrt{2} \int_{x_h^-}^{x_h^+} \frac{dx}{\sqrt{h - V(x)}}.$$

Since $x \mapsto \frac{1}{\sqrt{h - V(x)}}$ has singularities at $x = x_h^{\pm}$, to obtain an expression of $T'(h)$ it is first necessary to make some transformations. We perform the change of variable given by $g(x) = \sqrt[2m]{h} \sin\theta$, which yields to

$$T(h) = \sqrt{2} \int_{x_h^-}^{x_h^+} \frac{dx}{\sqrt{h - V(x)}} = \sqrt{2} h^{\frac{1-m}{2m}} \int_{-\pi/2}^{\pi/2} \left(g^{-1}\right)' \left(\sqrt[2m]{h} \sin\theta \right) \frac{\cos\theta \, d\theta}{\sqrt{1 - \sin^{2m}\theta}}.$$

Note that the last integral above is not improper because $\frac{\cos\theta}{\sqrt{1-\sin^{2m}\theta}}$ tends to $\frac{1}{\sqrt{m}}$ as $\theta \to \pm\pi/2$. Hence we can differentiate with respect to h and we obtain

$$\left(h^{\frac{m-1}{2m}} T(h)\right)' = \frac{1}{\sqrt{2}m} h^{\frac{1}{2m}-1} \int_{-\pi/2}^{\pi/2} \left(g^{-1}\right)'' \left(\sqrt[2m]{h} \sin\theta \right) \frac{\cos\theta \sin\theta \, d\theta}{\sqrt{1 - \sin^{2m}\theta}},$$

so that

$$T'(h) = \frac{1-m}{2mh} T(h) + \frac{1}{\sqrt{2}m} h^{\frac{2-3m}{2m}} \int_{-\pi/2}^{\pi/2} \left(g^{-1}\right)'' \left(\sqrt[2m]{h} \sin\theta \right) \frac{\cos\theta \sin\theta \, d\theta}{\sqrt{1 - \sin^{2m}\theta}}.$$

Now undoing the previous change of variable and taking $(g^{-1})''(z) = -(\frac{g''}{g'^3})(g^{-1}(z))$ into account, we get

$$T'(h) = \frac{1-m}{2mh} T(h) - \frac{1}{\sqrt{2}mh} \int_{x_h^-}^{x_h^+} \left(\frac{gg''}{g'^2}\right)(x) \frac{dx}{\sqrt{h - V(x)}}.$$

Finally, using that $g(x)^{2m} = V(x)$, we obtain

$$\begin{aligned} T'(h) &= \frac{1-m}{2mh} T(h) - \frac{1}{\sqrt{2}mh} \int_{x_h^-}^{x_h^+} \frac{2mV'V - (2m-1)V'^2}{V'^2}(x) \frac{dx}{\sqrt{h - V(x)}} \\ &= \frac{1}{h}\left(\frac{1-m}{2m} \int_{\gamma_h} \frac{dx}{y} - \frac{1}{2m} \int_{\gamma_h} \frac{2mV'V - (2m-1)V'^2}{V'^2}(x) \frac{dx}{y}\right) \\ &= \frac{1}{h} \int_{\gamma_h} \frac{V'^2 - 2V'V}{2V'^2}(x) \frac{dx}{y} = \frac{1}{h} \int_{\gamma_h} \frac{\mu_1(x)}{y}\, dx. \end{aligned}$$

This proves the case $k = 1$ of the first equality in the statement. Suppose now that the equality holds for $k = n$. Then

$$\begin{aligned} T'(h) &= \frac{1}{h^n} \int_{\gamma_h} \mu_n(x) y^{2n-3}\, dx = \frac{1}{h^{n+1}} \int_{\gamma_h} \left(\frac{1}{2}y^2 + V(x)\right) \mu_n(x) y^{2n-3}\, dx \\ &= \frac{1}{h^{n+1}} \int_{\gamma_h} \left(\frac{1}{2}\mu_n(x) + \frac{1}{2n-1}\left(\frac{\mu_n V}{V'}\right)'(x)\right) y^{2n-1}\, dx = \frac{1}{h^{n+1}} \int_{\gamma_h} \mu_{n+1}(x) y^{2n-1}\, dx, \end{aligned}$$

where in the third equality we apply Lemma 2.5 with $g = V\mu_n$ and $k = 2n - 1$. This shows the result for $k = n + 1$ and completes the proof of the first equality. The second equality in the statement is also easy to verify because, taking any function f,

$$\begin{aligned} \int_{\gamma_h} f(x) y^{2k-3}\, dx &= 2 \int_{x_h^-}^{x_h^+} f(x)\big(h - V(x)\big)^{\frac{2k-3}{2}}\, dx \\ &= 2\left(\int_{x_h^-}^{0} f(x)\big(h - V(x)\big)^{\frac{2k-3}{2}}\, dx + \int_{0}^{x_h^+} f(x)\big(h - V(x)\big)^{\frac{2k-3}{2}}\, dx\right) \end{aligned}$$

$$= 2\left(\int_{x_h^+}^{0} f\big(\sigma(x)\big)\big(h - V\big(\sigma(x)\big)\big)^{\frac{2k-3}{2}} \sigma'(x)\,dx + \int_0^{x_h^+} f(x)\big(h - V(x)\big)^{\frac{2k-3}{2}}\,dx\right)$$

$$= 2\int_0^{x_h^+} \mathscr{P}_\sigma(f)(x)\big(h - V(x)\big)^{\frac{2k-3}{2}}\,dx.$$

In the third equality above we took $V(x_h^+) = V(x_h^-) = h$ into account to conclude that $\sigma(x_h^+) = x_h^-$.

In order to show the second part of the result, let n be the number of zeros of $\mathscr{P}_\sigma(\mu_k)$ on $(0, x_r)$, counted with multiplicities, and assume that $k > n$. Then, by applying Proposition 2.2, there exist $g_0, g_1, \ldots, g_{n-1}$ analytic σ-even functions on (x_ℓ, x_r) such that $(g_0, g_1, \ldots, g_{n-1}, \mathscr{P}_\sigma(\mu_k))$ is an ECT-system on $(0, x_r)$. Thus, since any ECT-system is a CT-system and $\mathscr{P}_\sigma(g_i) = g_i$, $(\mathscr{P}_\sigma(g_0), \mathscr{P}_\sigma(g_1), \ldots, \mathscr{P}_\sigma(g_{n-1}), \mathscr{P}_\sigma(\mu_k))$ is a CT-system on $(0, x_r)$. Define

$$I_i(h) = \int_{\gamma_h} g_i(x) y^{2k-3}\,dx \quad \text{for } i = 0, 1, \ldots, n-1, \quad \text{and} \quad I_n(h) = \int_{\gamma_h} \mathscr{P}_\sigma(\mu_k)(x) y^{2k-3}\,dx.$$

Then, due to $k \geqslant n + 1$, by Theorem 2.4 we can assert that $(I_0, I_1, \ldots, I_n)$ is an ECT-system on $(0, h_0)$. In particular, taking $I_n(h) = h^k T'(h)$ into account, it follows that $T'(h)$ has at most n zeros on $(0, h_0)$ counted with multiplicities. This completes the proof of the result. □

Finally we show a generalization of an Opial's result (see [17]) that will be used in the applications.

Theorem 2.6. *If the center at the origin of system* (1) *has a monotone period function then* $x \mapsto \frac{V(x)}{(x-\sigma(x))^2}$ *is monotone on* $(0, x_r)$.

Proof. Opial's result (see Theorem 6 in [17]) asserts that if the period function of the center of system (1) is monotone and V is an *even function* then $x \mapsto \frac{V(x)}{x^2}$ is monotone on $(0, x_r)$. (Although Opial states his result for global centers, one can check that this hypothesis is not necessary.) In order to generalize this result to potentials with V not being an even function, we will use a result from [13]. To state this result we need a different parametrization of the period function. Thus, for any $x \in (0, x_r)$, $\widehat{T}_V(x)$ will denote the period of the periodic orbit of system (1) passing through $(x, 0) \in \mathbb{R}^2$. Then, with this notation and setting

$$\overline{V} := V \circ \kappa^{-1} \quad \text{with } \kappa(x) = \frac{x - \sigma(x)}{2},$$

Theorem 3 in [13] shows that $\widehat{T}_V(x) = \widehat{T}_{\overline{V}}(\kappa(x))$ for all $x \in (0, x_r)$. The key point is that, on account of $\kappa(\sigma(x)) = -\kappa(x)$, $\overline{V}$ is an even function and so we can apply Opial's result. (We remark that the result in [13] is only stated for non-degenerate centers, i.e., the case $m = 1$, but it can be shown that this assumption is not necessary.)

Now, by assumption, the period function of the center associated to the potential system given by V is monotone. This implies, on account of the result in [13], that so it is the one associated to the potential system given by $\overline{V}$. The latter is an even function, so by Opial's result we can assert that $x \mapsto \frac{\overline{V}(x)}{x^2}$ is monotone. Since $\kappa(x) = \frac{x-\sigma(x)}{2}$ is a diffeomorphism and $\overline{V} = V \circ \kappa^{-1}$, this proves the result. $\square$

Remark 2.7. The necessary condition in Theorem 2.6 has an interesting geometric interpretation in terms of the length of the projection of the periodic orbits. If $g(x) = \operatorname{sgn}(x)\sqrt[2m]{V(x)}$, then it follows that

$$\ell(h) := x_h^+ - x_h^- = g^{-1}\big(\sqrt[2m]{h}\big) - g^{-1}\big(-\sqrt[2m]{h}\big)$$

and $\sigma(x) = g^{-1}(-g(x))$. Define $S(x) = \frac{V(x)}{(x-\sigma(x))^2}$ and note that then

$$S\big(g^{-1}(x)\big) = \frac{V(g^{-1}(x))}{(g^{-1}(x) - g^{-1}(-x))^2} = \frac{x^{2m}}{\ell(x^{2m})^2}.$$

Hence $S(g^{-1}(\sqrt[2m]{h})) = \frac{h}{\ell(h)^2}$, and so it is clear that S is monotone if and only if so it is $h \mapsto \frac{\ell(h)}{\sqrt{h}}$. Accordingly, on account of Theorem 2.6, if the center at the origin of system (1) has a monotone period function then $h \mapsto \frac{\ell(h)}{\sqrt{h}}$ is monotone. Interestingly enough, Theorem A in [5] asserts that the center is isochronous (i.e., it has a constant period function) if and only if $h \mapsto \frac{\ell(h)}{\sqrt{h}}$ is constant.

3. Applications

In this section we shall often apply Sturm's Theorem and we will use the properties of the resultant between two polynomials. Moreover, in order to study the zeros of a polynomial depending on a parameter, we shall use the notion of discriminant. The interested reader is referred to Appendix A for details about these tools. We shall also take the following easy observation into account.

Remark 3.1. If V is even in the usual sense (i.e., $\sigma = -\mathrm{Id}$), then so it is μ_k and, hence, $\mathscr{P}_\sigma(\mu_k) = \mu_k$.

3.1. A pendulum

Lichardová studies in [12] the potential system associated to

$$V(x) = -\gamma\cos x + \frac{1}{2}\cos^2 x + \gamma - \frac{1}{2}, \tag{2}$$

which models a motion of a pendulum rotating about its vertical axis. Due to the periodicity of $x \mapsto \cos x$, it suffices to study the system on the vertical strip $[-\pi, \pi] \times \mathbb{R}$. It can be easily shown that there is a non-degenerate center at the origin for $\gamma \geqslant 1$. Lichardová proves in Theorem 1 that its period function is monotone for $\gamma \geqslant 4$ and that it has a unique critical period for $\gamma \in (1, 4)$.

The projection on the x-axis of the period annulus of the center at the origin is $(-\pi, \pi)$. Thus, since V is an even function, from Remark 3.1 it follows that to apply Theorem A we must study

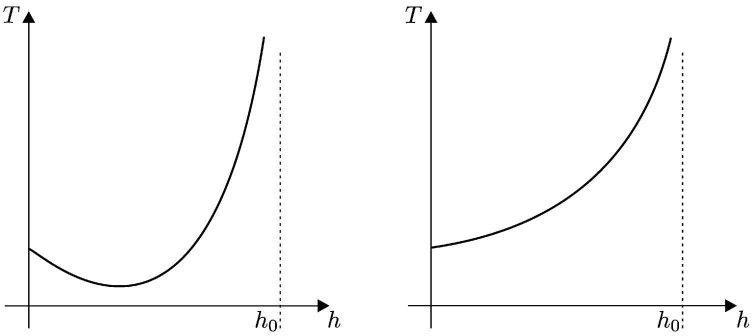

Fig. 1. Period function of system (2) for $\gamma \in (1,4)$ (left) and $\gamma \geqslant 4$ (right).

the number of zeros of μ_2 in $(0,\pi)$. Straightforward computations show that $4V'(x)^4\mu_2(x) = -(1-\cos x)^2 p(\cos x)$ with

$$p(x) = 2x^4 - 4(2\gamma-1)x^3 + \big(4\gamma^2-1\big)x^2 - \big(\gamma^3+6\gamma^2-11\gamma+6\big)x + 2\gamma^3 - 4\gamma^2 + 6\gamma - 3.$$

Accordingly the problem is then to compute the number of zeros of the polynomial p on $(-1,1)$. Let us denote this number by $\mathcal{Z}(\gamma)$. As we explain in Appendix A, $\mathcal{Z}(\gamma)$ can only change at those parameters γ such that the discriminant of p, say $\mathscr{D}(\gamma)$, is zero or either $p(1) = (\gamma-4)(\gamma-1)^2$ or $p(-1) = 3\gamma(\gamma+1)^2$ vanish. The discriminant of p (i.e., the resultant between p and p') is

$$\begin{aligned}\mathscr{D}(\gamma) = \gamma\big(&41\gamma^7 - 696\gamma^6 + 9558\gamma^5 - 36828\gamma^4 + 65505\gamma^3 \\ &- 53604\gamma^2 + 16318\gamma - 24\big)\big(\gamma^2-1\big)^2,\end{aligned}$$

which one can prove by applying Sturm's Theorem that it does not vanish for $\gamma > 1$. Therefore $\mathcal{Z}(\gamma)$ for $\gamma > 1$ can only change at $\gamma = 4$. By using Sturm's Theorem once again it follows that $\mathcal{Z}(2) = 1$ and $\mathcal{Z}(4) = \mathcal{Z}(5) = 0$. Hence p does not vanish on $(-1,1)$ for $\gamma \geqslant 4$ and it vanishes exactly once for $\gamma \in (1,4)$. Thus, by Theorem A, the period function of the center at the origin of the potential system associated to (2) is monotone for $\gamma \geqslant 4$ and it has exactly one critical period for $\gamma \in (1,4)$. See Fig. 1 where we set $h_0 = V(\pi)$.

3.2. A two-monomial family

Let us consider the potential system associated to

$$V(x) = \frac{a}{n+1}x^{n+1} + \frac{b}{m+1}x^{m+1}, \tag{3}$$

where m and n are *natural odd numbers* with $n < m$ and $a > 0$. This system has a center at the origin that is degenerate for $n \geqslant 3$ and non-degenerate for $n = 1$. If the center is global then (see [7] for instance) it has a monotone decreasing period function. In this example we consider the non-global case. It is clear that then the center is bounded, and this implies that the outer boundary of $\mathcal{P}$ has singular points of the system. By means of a reparametrization we can assume without loss of generality that this singular point is located at $(1,0)$, so that $b = -a$ and, since V is an even function, the projection of $\mathcal{P}$ on the x-axis is $(-1,1)$. In this case one can verify that $4V'(x)^4\mu_2(x)=a^4x^{4n}p(x^{m-n})$ with

$$\begin{aligned}p(x)=&-\frac{(m+3)(m-1)}{(m+1)^2}x^4+4\frac{4m^2-n+3n^2+4m^2n-4mn-3mn^2-3+n^3-m^3}{(m+1)^2(n+1)}x^3\\&+2\left(\frac{6m+21m^2n^2-13m^2+20mn-13n^2+6n+9}{(n+1)^2(m+1)^2}\right.\\&\left.-\frac{16m^3n+6n^3+6m^3-4n^4-4m^4+16n^3m}{(n+1)^2(m+1)^2}\right)x^2\\&+4\frac{m^3-3m^2n+3m^2-4mn^2-m-4mn-n^3+4n^2-3}{(m+1)(n+1)^2}x-\frac{(n+3)(n-1)}{(n+1)^2}.\end{aligned}$$

The problem is now, recall Remark 3.1, to compute the number of zeros of p in $(0,1)$. In principle this is a very difficult study because p is a two-parametric polynomial of fourth degree. However, curiously enough, it turns to be trivial if we perform the transformation $y=f(x):=\frac{1}{x}-1$, which sends the interval $(0,1)$ to $(0,+\infty)$. Indeed, one can verify that

$$p\big(f^{-1}(y)\big)=\frac{\alpha_0+\alpha_1y+\alpha_2y^2+\alpha_3y^3+\alpha_4y^4}{(m+1)^2(n+1)^2(y+1)^4}$$

with

$$\begin{aligned}\alpha_0&=12(m-n)^4, &\alpha_3&=4(m+1)(mn-2)(m-n+1)(m-n),\\\alpha_1&=4(7m+5-5n)(m-n)^3, &\alpha_4&=-(n+3)(n-1)(m+1)^2.\\\alpha_2&=4(5m-2n+4)(m+1-n)(m-n)^2,\end{aligned}$$

Since $m>n$, the sign of coefficients of the polynomial in the numerator is $\{+,+,+,+,-\}$ for $n>1$ and $\{+,+,+,+\}$ for $n=1$. Thus, by the Descartes' Rule, $p\circ f^{-1}$ has exactly one positive root for $n>1$ and it does not vanish on $(0,+\infty)$ for $n=1$. Consequently, on account of $f((0,1))=(0,+\infty)$, we can assert that p has exactly one root on $(0,1)$ for $n>1$ and that it does not vanish on $(0,1)$ for $n=1$. Hence, by applying Theorem A, the period function of the center at the origin of the system associated to (3) is monotone for $n=1$ and it has exactly one critical period for $n>1$ (see Fig. 2).

3.3. A rational family

The next example comes from the study of the potential system associated to

$$V(x)=\frac{x^m}{ax^n+1}\tag{4}$$

with m and n *even natural numbers.* This system has a center at the origin, non-degenerate for $m=2$ and degenerate for $m\geqslant 4$. Sabatini [19], as an example of application of his convexity criterion, studies the case $m=n=4$ and $a=1$. Setting

$$\mathscr{B}(m,n):=n^2+2mn-2m^2-2n+4m+1,$$

we can prove:

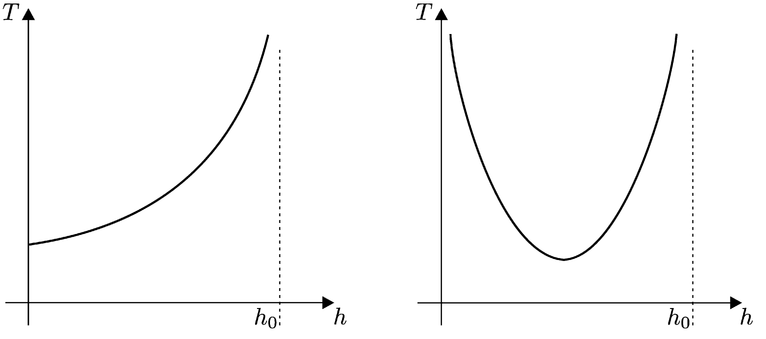

Fig. 2. Period function of system (3) for $n = 1$ (left) and $n \geqslant 3$ (right).

Proposition 3.2. *Consider the period function of the center at the origin of the potential system associated to* (4) *and assume that m and n are even natural numbers. If $a < 0$ then the period function is monotone decreasing. In case that $a > 0$ then the following hold*:

(a) *If $m \leqslant n$, then the period function has a unique critical period* (*a minimum*) *when $m \geqslant 4$, and it is monotone increasing when $m = 2$.*
(b) *If $m > n$, then the period function is monotone decreasing in case that $\mathscr{B}(m, n) < 0$.*

Proof. If $a < 0$ then the projection of $\mathcal{P}$ on the x-axis is $\mathcal{I} = (-\bar{x}, \bar{x})$ with $\bar{x}^n = -1/a$. If $a > 0$ then the differential system is defined in the whole plane and the projection of $\mathcal{P}$ depends on the sign of $\kappa := m - n$; $\mathcal{I} = (-\infty, +\infty)$ in case that $\kappa \geqslant 0$ and $\mathcal{I} = (-\bar{x}, \bar{x})$, with $\bar{x}^n = -\frac{m}{a\kappa}$, in case that $\kappa < 0$.

Let us prove (a) first, so assume that $a > 0$ and $\kappa \leqslant 0$. Some calculations show that

$$\mu_2(x) = \frac{p(ax^n)}{4(a\kappa x^n + m)^4},$$

where $p(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 x^4$ with

$$\begin{aligned}
\beta_0 &= (\kappa + n)^2\big(4 - (\kappa + n)^2\big),\\
\beta_1 &= 4(\kappa + n)\big(n^3 + 2n + 3n^2 + 4\kappa - \kappa^3 - 2\kappa^2 n - \kappa n^2\big),\\
\beta_2 &= 12n^3 - 6\kappa^4 - 12\kappa^3 n - 6\kappa^2 n^2 + 8n^4 + 24\kappa^2 + 24\kappa n + 4n^2 + 24\kappa n^2,\\
\beta_3 &= 4\kappa\big(4\kappa + 2n + 3n^2 - \kappa^3 - \kappa^2 n - n^3\big),\\
\beta_4 &= \kappa^2\big(4 - \kappa^2\big).
\end{aligned}$$

Let us consider the case $\kappa = 0$ (i.e., $m = n$) first. In this case the sign of the coefficients of p is $\{-, +, +\}$ if $m = n \geqslant 4$ and $\{+, +\}$ if $m = n = 2$. So by applying Descartes' Rule to p, μ_2 has exactly one zero on $(0, +\infty)$ in the first case and it does not vanish on $(0, +\infty)$ in the second one. By Theorem A this proves (a) for $\kappa = 0$. Next we claim that if $\kappa < 0$ then p has one zero on $(0, -\frac{m}{\kappa})$ for $m \geqslant 4$ and it does not vanish there for $m = 2$. This, on account of Theorem A, will finish the proof of (a). To this end we perform the transformation $y = f(x) := \frac{m}{x} + \kappa$, that sends the interval $(0, -\frac{m}{\kappa})$ to $(0, +\infty)$. One can verify that

$$p\big(f^{-1}(y)\big) = \frac{m^2}{(y-\kappa)^4}\big(\gamma_0 + \gamma_1 y + \gamma_2 y^2 + \gamma_3 y^3 + \gamma_4 y^4\big),$$

where

$$\begin{aligned} \gamma_0 &= 12\kappa^2 n^4, & \gamma_3 &= 4n(n+2)(n+1), \\ \gamma_1 &= -4\kappa n^3(5n+3-2\kappa), & \gamma_4 &= 4-m^2. \\ \gamma_2 &= 4n^2(n+1)(2n+1-3\kappa), \end{aligned}$$

Since $\kappa < 0$, the sign of the coefficients above is $\{+,+,+,+,-\}$ for $m \geqslant 4$ and $\{+,+,+,+\}$ for $m = 2$, and so the claim follows by using Descartes's Rule.

For the remaining cases we must show monotonicity, so it suffices to consider μ_1. It follows that

$$\mu_1(x) = \frac{q(ax^n)}{2(\kappa a x^n + m)^2},$$

where q is the quadratic polynomial given by

$$q(x) = \kappa(2-\kappa)x^2 + 2\big(n^2 + 2\kappa + n - \kappa^2 - \kappa n\big)x - m(m-2).$$

If $a < 0$ then we must verify that $q(x) < 0$ for $x \in (-1, 0)$. Since this is very easy, for the sake of shortness it is left to the reader. The assertion in (b) (the case $\kappa > 0$ and $a > 0$) follows showing that $q(x) < 0$ for $x > 0$. This is obvious because one can check that the discriminant of q is precisely $4n^2\mathscr{B}(m,n)$. $\square$

3.4. *A four-monomial example*

As an application of our results we want to discuss two more situations. In the first one we treat the case of a center with a period function that either it is monotone or it has an even number of critical periods. In the second one we want to show a case in which the center has a unique critical period but the period function $h \mapsto T(h)$ is not convex. Both situations occur in 1-parameter subfamilies of the potential systems with

$$V(x) = \frac{1}{2}x^2 + \frac{a}{4}x^4 + \frac{b}{6}x^6 + \frac{c}{8}x^8. \tag{5}$$

In the discussion bellow we will use that the first period constant of the center at the origin is $-a$ (see for instance [8]).

Example 3.3. The first situation described above comes from taking a subfamily in which $\mathcal{P}$ is bounded (so that the period function tends to infinity as we approach to the outer boundary of $\mathcal{P}$) and the first period constant is positive. To this end we take system (5) with $a = -1$ and $c = -b$. The period annulus of this subfamily is bounded and its projection on the x-axis is $(-1, 1)$ for all $b > -1$.

Some calculations show that $24V'(x)^6\mu_3(x) = x^8 p(x^2)$ where p is a polynomial of degree 17 in x (that depends on the parameter b). In order to study its zeros for $x \in (0,1)$ we first note that

$$p(1) = 120(b+6)^3(b+1)^3 \quad \text{and} \quad p(0) = 62208,$$

so that the number of zeros of p in $(0,1)$ for $b > -1$ can only change due to the vanishing of its discriminant (since it is not possible the emergence or disappearance of zeros at $x = 0$ or $x = 1$). The discriminant, which is of course a polynomial in b, factorizes as

$$\mathscr{D}(b) = b^{138}(b+1)^{12}(b+6)^{36}(64b+9)^{36}\ell(b)^2,$$

where ℓ is a polynomial of degree 29. By applying Sturm's Theorem it follows that ℓ has exactly two roots for $b \in (-1,+\infty)$, say b_1 and b_2, and one can show that $b_1 \approx -0.194$ and $b_2 \approx 2.391$. Accordingly the number of zeros of p on $(0,1)$ for $b > -1$, say $\mathcal{Z}(b)$, can only change at $b = b_1$, $b = -9/64 \approx -0.140$, $b = 0$ and $b = b_2$.

Now, since by applying Sturm's Theorem once again we have that $\mathcal{Z}(-\frac{1}{2}) = 0$, taking the above discussion into account, we can assert that $\mathcal{Z}(b) = 0$ for any $b \in (-1, b_1)$. Using the same argument we conclude that $\mathcal{Z}(b) = 0$ for $b \in (b_1, -\frac{9}{64}) \cup (-\frac{9}{64}, 0) \cup (0, b_2)$ and that $\mathcal{Z}(b) = 2$ for $b > 0$. By applying Theorem A this shows that the period function is monotone increasing when $b \in (-1, b_2) \setminus \{b_1, -\frac{9}{64}, 0\}$ and that it has either 0 or 2 critical periods for when $b \in (b_2, +\infty)$.

Now we compute μ_1 in order to be more precise. It follows that $24x^4 V'(x)^2\mu_1(x) = q(x^2)$ with

$$q(x) = -9b^2x^5 + 19b^2x^4 - \bigl(27b + 8b^2\bigr)x^3 + 81bx^2 - (40b+6)x + 18.$$

Studying this polynomial with the same bifurcation arguments as before and applying Theorem A, we conclude that the period function is monotone increasing when $b \in (-1, b_3)$, where $b_3 \approx 2.843$ is the positive root of $73000b^3 + 2511717b^2 - 7701102b - 87723 = 0$. In particular this shows that for $b \in (b_2, b_3)$ we do not have 2 critical periods but monotonicity. Finally, to study what happens for $b > b_3$, we use Opial's necessary condition of monotonicity. Note that

$$\left(\frac{V'(x)}{x^2}\right)' = -\frac{x}{12}\bigl(6 - 8bx^2 + 9bx^4\bigr).$$

It is very easy to show that the above function has two roots on $(0,1)$ for $b > \frac{27}{8} = 3.375$. Accordingly the period function is not monotone for these parameters. In short, we have proved that the period function of system (5) with $a = -1$ and $c = -b$ is monotone increasing for $b \in (-1, b_3)$, it has either 0 or 2 critical periods for $b \in (b_3, \frac{27}{8})$ and it has exactly two critical periods for $b > \frac{27}{8}$. Fig. 3, where we set $h_0 = V(1)$, gathers this result.

Example 3.4. The second example described above follows from taking a subfamily of system (5) for which the center is global (because then the period function tends to zero as $h \mapsto +\infty$) and such that the period function begins increasing. It is clear that in this situation there exists at least one critical period but the period function $h \mapsto T(h)$ cannot be convex.

Taking the above discussion into account, we consider the 1-parameter subfamily of centers in (5) with $b = c = 1$ and $a < 0$. Straightforward computations show that the center is global

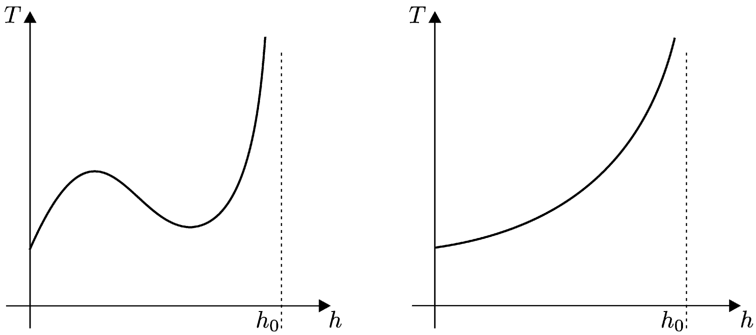

Fig. 3. Period function for $b \in (-1, b_3)$ with $b_3 \approx 2.843$ (right) and $b > \frac{27}{8} = 3.375$ (left).

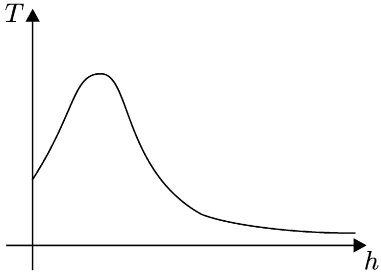

Fig. 4. Period function for $a \in (a_1, 0)$ with $a_1 \approx -0.668$.

if and only if $a > a^\star$, where $a* \approx -2.610$ is the unique real root of $4a^3 - a^2 - 18a + 31 = 0$. Accordingly if $a \in (a^\star, 0)$ then the period function is not convex but it has at least one critical period.

To bound the number of critical periods we compute μ_2. It turns out that $4V'(x)^4\mu_2(x) = x^6 p(x^2)$, where p is a polynomial of degree 11 in x (that depends on the parameter a). In order to study the number of zeros of p in $(0, +\infty)$, say $\mathscr{Z}(a)$, we first note that its leading coefficient is -135 (i.e., it does not depend on a) and that $p(0) = -864a$. Thus, if $\mathscr{D}(a)$ denotes the discriminant of p, we can assert that $\mathscr{Z}(a)$ for $a \in (a^\star, 0)$ can only change at those values of a such that $\mathscr{D}(a) = 0$. By Sturm's Theorem, $\mathscr{D}(a) = 0$ has only one root in $(a^\star, 0)$, say a_1, and it is easy to show that $a_1 \approx -0.668$. Consequently, since one can verify (by applying Sturm's Theorem once again) that $\mathscr{Z}(-2) = 3$ and $\mathscr{Z}(-\frac{1}{2}) = 1$, we conclude that $\mathscr{Z}(a) = 3$ for $a \in (a^\star, a_1)$ and $\mathscr{Z}(a) = 1$ for $a \in (a_1, 0)$. By Theorem A, the center of system (5) with $b = c = 1$ and $a \in (a_1, 0)$ has exactly one critical period but its period function is not convex (see Fig. 4).

3.5. Cubic potentials

Our last application comes from the papers of Chow and Sanders [4] and Gavrilov [10], where the period functions of the cubic potential systems are studied. More concretely, Gavrilov proves that the period functions of $H(x, y) = \frac{1}{2}y^2 + V(x)$ with V being a polynomial of fourth degree can have at most one critical period, and that if the critical period exists then the center is global. To illustrate the application of our result we shall focus on the case in which the period function is not monotone.

Any cubic potential system with a global center can be brought to the one associated to

$$V(x) = \frac{1}{2}x^2 + \frac{a}{3}x^3 + \frac{1}{4}x^4 \quad \text{with } a \in (-2, 2).$$

The first period constant of the center is $10a^2 - 9$ (see [8]) and, since the center is global, the period function tends to zero as $h \to +\infty$. Thus, from [4,10], the period function is monotone for $a \in [\frac{-3}{\sqrt{10}}, \frac{3}{\sqrt{10}}]$ and it has exactly one critical period for $a \in (-2, \frac{-3}{\sqrt{10}}) \cup (\frac{3}{\sqrt{10}}, 2)$.

Some calculations show that

$$\mu_2(x) = -\frac{x}{144} \frac{q(x)}{(1 + ax + x^2)^4}$$

with

$$\begin{aligned} q(x) = {} & 27x^7 + 108ax^6 + \big(108 + 162a^2\big)x^5 + 2a\big(171 + 52a^2\big)x^4 \\ & + \big(189 + 348a^2 + 20a^4\big)x^3 + 16a\big(27 + 5a^2\big)x^2 + \big(120a^2 + 216\big)x + 72a. \end{aligned}$$

To apply Theorem A it is necessary to compute the number of zeros of $\mathscr{P}_\sigma(\mu_2)$ on $(0, +\infty)$, where recall that $\sigma \neq \mathrm{Id}$ is the involution defined by means of $V(x) = V(\sigma(x))$. Taking $V'(x) = V'(\sigma(x))\sigma'(x)$ into account, it is clear that it suffices to study $\mu_2(x)V'(\sigma(x)) - \mu_2(\sigma(x))V'(x) = 0$, i.e., the common roots of

$$S_1(x, z) = \mu_2(x)V'(z) - \mu_2(z)V'(x) \quad \text{and} \quad S_2(x, z) = V(x) - V(z).$$

One can verify that

$$S_1(x, z) = \frac{xz(x - z)\widehat{S}_1(x, z)}{(1 + ax + x^2)^4(1 + az + z^2)^4} \quad \text{and} \quad S_2(x, z) = \frac{x - z}{12}\widehat{S}_2(x, z),$$

where $\widehat{S}_i$ are polynomials (for instance, $\widehat{S}_2(x, z) = 3x^3 + (4a + 3z)x^2 + (3z^2 + 4az + 6)x + 3z^3 + 6z + 4az^2$). Since $\sigma \neq \mathrm{Id}$, we compute the resultant (with respect to z) between $\widehat{S}_1$ and $\widehat{S}_2$. This is a polynomial in x, say $\mathscr{R}(x)$, that depends on the parameter $a \in (-2, 2)$. For the sake of shortness we do not write $\mathscr{R}$ here (it has degree 50). The essential information is that

$$\mathscr{R}(0) = 7776\big(10a^2 - 9\big)\big(16a^4 - 72a^2 - 27\big)\big(2a^2 - 9\big)^4$$

and that its leading coefficient is 387 420 489, so that it does not vanish. As usual, let us denote by $\mathcal{Z}(a)$ the number of roots of $\mathscr{R}(x) = 0$ for $x \in (0, +\infty)$ and by $\mathscr{D}(a)$ the discriminant of $\mathscr{R}(x)$. Since neither $16a^4 - 72a^2 - 27$ or $2a^2 - 9$ vanishes on $(-2, 2)$, the value of $\mathcal{Z}(a)$ for $a \in (-2, 2)$ can only change at those parameters a such that $(10a^2 - 9)\mathscr{D}(a) = 0$. Of course $\mathscr{D}(a)$ is a huge polynomial, but by means of Sturm's Theorem we can compute and approximate its zeros on $(-2, 2)$. They are $a = 0$ and $a = \pm\hat{a}_i$, $i = 1, 2, \ldots, 5$ where

$$\hat{a}_1 \approx 0.728, \quad \hat{a}_2 \approx 0.889, \quad \hat{a}_3 = \frac{3}{\sqrt{10}} \approx 0.949, \quad \hat{a}_4 \approx 1.64, \quad \hat{a}_5 \approx 1.88.$$

(Note that $a = \pm\hat{a}_3$ are the zeros of $10a^2 - 9$.) Taking one concrete parameter in each interval and applying Sturm's Theorem we get that

$$\begin{array}{lll}
\mathcal{Z}(a) = 1 \quad \text{on } (-2, -\hat{a}_5), & \mathcal{Z}(a) = 0 \quad \text{on } (-\hat{a}_1, 0), & \mathcal{Z}(a) = 3 \quad \text{on } (\hat{a}_4, \hat{a}_5), \\
\mathcal{Z}(a) = 1 \quad \text{on } (-\hat{a}_5, -\hat{a}_4), & \mathcal{Z}(a) = 0 \quad \text{on } (0, \hat{a}_1), & \mathcal{Z}(a) = 3 \quad \text{on } (\hat{a}_5, 2). \\
\mathcal{Z}(a) = 1 \quad \text{on } (-\hat{a}_4, -\hat{a}_3), & \mathcal{Z}(a) = 0 \quad \text{on } (\hat{a}_1, \hat{a}_2), & \\
\mathcal{Z}(a) = 0 \quad \text{on } (-\hat{a}_3, -\hat{a}_2), & \mathcal{Z}(a) = 0 \quad \text{on } (\hat{a}_2, \hat{a}_3), & \\
\mathcal{Z}(a) = 0 \quad \text{on } (-\hat{a}_2, -\hat{a}_1), & \mathcal{Z}(a) = 1 \quad \text{on } (\hat{a}_3, \hat{a}_4), &
\end{array}$$

Moreover $\mathcal{Z}(\frac{\pm 3}{\sqrt{10}}) = \mathcal{Z}(0) = 0$. By applying Theorem A, if $a \in [\frac{-3}{\sqrt{10}}, \frac{3}{\sqrt{10}}] \setminus \{\pm\hat{a}_1, \pm\hat{a}_2\}$, then the period function is monotone, and if $a \in (-2, \frac{-3}{\sqrt{10}}) \cup (\frac{3}{\sqrt{10}}, \hat{a}_4) \setminus \{-\hat{a}_5, -\hat{a}_4\}$, then it has exactly one critical period.

(In fact one can easily show that $\mathscr{P}_\sigma(\mu_1)$ does not vanish on $(0, +\infty)$ for $a \in [\frac{-3}{\sqrt{10}}, \frac{3}{\sqrt{10}}]$, so that, by applying Theorem A, we get the monotonicity for $a = \pm\hat{a}_1$ and $a = \pm\hat{a}_2$ as well. In order to prove the result for $a \in (\hat{a}_4, 2)$ it would be necessary to study $\mathscr{P}_\sigma(\mu_k)$ with $k \geqslant 3$.)

Appendix A

A.1. Resultant of two polynomials

Given two polynomials $p, q \in \mathbb{C}[x]$, say

$$\begin{aligned}
p(x) &= a_0 x^m + a_1 x^{m-1} + \cdots + a_m \quad \text{with } a_0 \neq 0, \\
q(x) &= b_0 x^n + b_1 x^{n-1} + \cdots + b_n \quad \text{with } b_0 \neq 0,
\end{aligned}$$

the *resultant* of p and q, denoted by $\operatorname{Res}(p, q)$ is the $(m+n) \times (m+n)$ determinant

$$\operatorname{Res}(p,q) = \det \begin{pmatrix}
a_0 & & & & b_0 & & & \\
a_1 & a_0 & & & b_1 & b_0 & & \\
a_2 & a_1 & \ddots & & b_2 & b_1 & \ddots & \\
\vdots & a_2 & \ddots & a_0 & \vdots & \ddots & & b_0 \\
a_m & \vdots & \ddots & a_1 & b_n & \vdots & \ddots & b_1 \\
 & a_m & & a_2 & & b_n & & b_2 \\
 & & \ddots & \vdots & & & \ddots & \vdots \\
 & & & a_m & & & & b_n
\end{pmatrix}$$

where the blank spaces are filled with zeros. When we want to emphasize the dependence on x we will write $\operatorname{Res}(p, q, x)$ instead of $\operatorname{Res}(p, q)$. The three basic properties of the resultant are (see [6]):

1. $\operatorname{Res}(p, q)$ is an integer polynomial in the coefficients of p and q.
2. $\operatorname{Res}(p, q) = 0$ if and only if p and q have a nontrivial common factor in $\mathbb{C}[x]$.
3. There are polynomials $A, B \in \mathbb{C}[x]$ such that $A(x)p(x) + B(x)q(x) = \operatorname{Res}(p, q)$. The coefficients of A and B are integer polynomials in the coefficients of p and q.

Resultants can be used to eliminate variables from systems of equations. As an example, let us suppose that we want to study

$$\begin{cases} xy-1=0, \\ x^2+y^2-4=0. \end{cases}$$

Here we have two variables to work with, but if we regard $p(x,y) := xy-1$ and $q(x,y) := x^2+y^2-4$ as polynomials in x whose coefficients are polynomials in y, we can compute the resultant with respect to x to obtain $\mathrm{Res}(p,q,x) = y^4-4y^2+1$. By the third property above, there are polynomials $A, B \in \mathbb{C}[x,y]$ such that $A(x,y)p(x,y)+B(x,y)q(x,y) = y^4-4y^2+1$. Accordingly, y^2-4y^2+1 vanishes at any common solution of $p=q=0$. Thus, by solving $y^4-4y^2+1=0$ we can find the y-coordinates of these solutions.

The *discriminant* of $p \in \mathbb{C}[x]$ is $\mathrm{Disc}(p) = \mathrm{Res}(p,p')$, where p' is the derivative of p. Consequently, p has no multiple roots if and only if $\mathrm{Disc}(p) \neq 0$. This property is very useful in case that p depends on some parameter μ and we want to study the number of its zeros on some interval I. To fix ideas let us suppose that $I=(0,1)$, the other cases (including $I=\mathbb{R}$) follow similarly. Denote the number of zeros of p_μ on $(0,1)$ by $\mathcal{Z}(\mu)$. Then, as the parameter μ varies, $\mathcal{Z}(\mu)$ can only change due to the collision of several zeros or the emergence (or disappearance) of some zero at the endpoints of $(0,1)$. The first type of bifurcation occurs at those parameters μ such that the discriminant of p_μ, say $\mathscr{D}(\mu)$, vanishes. The second one when either $p_\mu(0)=0$ or $p_\mu(1)=0$. Thus $\mathscr{D}(\mu)p_\mu(0)p_\mu(1)=0$ splits up the parameter space in several connected components and $\mathcal{Z}(\mu)$ is constant in each one.

A.2. Sturm's Theorem

A sequence $\{f_0, f_1, \dots, f_m\}$ of continuous real functions on $[a,b]$ is called a *Sturm's sequence* for $f=f_0$ on $[a,b]$ if the following is verified:

1. f_0 is differentiable on $[a,b]$.
2. f_m does not vanish on $[a,b]$.
3. If $f(x_0)=0$ with $x_0 \in [a,b]$ then $f_1(x_0)f_0'(x_0)>0$.
4. If $f_i(x_0)=0$ with $x_0 \in [a,b]$ then $f_{i+1}(x_0)f_{i-1}(x_0)<0$.

Sturm's Theorem. *Let $\{f_0, f_1, \dots, f_m\}$ be a Sturm's sequence for $f=f_0$ on $[a,b]$ with $f(a)f(b) \neq 0$. Then the number of roots of f on (a,b) is equal to $V(a)-V(b)$, where $V(c)$ is the number of changes of sign in the sequence $\{f_0(c), f_1(c), \dots, f_m(c)\}$.*

There is a simple procedure to construct a Sturm's sequence in case that f is polynomial. Indeed, if $p(x)$ is a polynomial of degree n then we define $\{p_0, p_1, \dots, p_m\}$ with $m \leqslant n$ setting $p_0=p$, $p_1=p'$ and

$$p_{i-1}(x) = q_i(x)p_i(x) - p_{i+1}(x), \quad \text{for } i=1,2,\dots,m-1,$$
$$p_{m-1}(x) = q_m(x)p_m(x),$$

where $q_i(x)$ and $p_{i+1}(x)$ are respectively the quotient and the remainder (the latter with the sign changed) of the division of $p_{i-1}(x)$ by $p_i(x)$. The construction of this sequence ends when the remainder is zero, i.e., $p_{m+1} = 0$. In this case, since this is essentially the Euclides' algorithm, p_m is the greatest common divisor of p_0 and p_1. If all the zeros of p are simple then p_m does not vanish and it is easy to show that $\{p_0, p_1, \dots, p_m\}$ is a Sturm's sequence for p on any interval. If p has zeros with multiplicity then p_m vanishes. Since p_m divides p_0 and p_1, it also divides p_i for $i = 2, 3, \dots, m$ and, setting $\bar{p}_i = p_i/p_m$, it follows that $\{\bar{p}_0, \bar{p}_1, \dots, \bar{p}_m\}$ is a Sturm's sequence for p on any interval.

References

[1] L. Bonorino, E. Brietzke, J. Lukaszczyk, C. Taschetto, Properties of the period function for some Hamiltonian systems and homogeneous solutions of a semilinear elliptic equation, J. Differential Equations 214 (2005) 156–175.

[2] C. Chicone, review in MathSciNet, ref. 94h:58072.

[3] C. Chicone, The monotonicity of the period function for planar Hamiltonian vector fields, J. Differential Equations 69 (1987) 310–321.

[4] S. Chow, J. Sanders, On the number of critical points of the period, J. Differential Equations 64 (1986) 51–66.

[5] A. Cima, F. Mañosas, J. Villadelprat, Isochronicity for several classes of Hamiltonian systems, J. Differential Equations 157 (1999) 373–413.

[6] D. Cox, J. Little, D. O'Shea, Using Algebraic Geometry, Grad. Texts in Math., vol. 185, Springer, 2005.

[7] E. Freire, A. Gasull, A. Guillamon, First derivative of the period function with applications, J. Differential Equations 204 (2004) 139–162.

[8] A. Gasull, A. Guillamon, V. Mañosa, An explicit expression of the first Liapunov and period constants with applications, J. Math. Anal. Appl. 211 (1997) 190–212.

[9] A. Gasull, A. Guillamon, J. Villadelprat, The period function for second-order quadratic ODEs is monotone, Qual. Theory Dyn. Syst. 4 (2004) 329–352.

[10] L. Gavrilov, Remark on the number of critical points of the period, J. Differential Equations 101 (1993) 58–65.

[11] M. Grau, F. Mañosas, J. Villadelprat, A Chebyshev criterion for Abelian integrals, preprint, 2008, arXiv:0805.1140v2 [math.DS].

[12] H. Lichardová, The period of a whirling pendulum, Math. Bohem. 126 (2001) 593–606.

[13] F. Mañosas, P.J. Torres, Two inverse problems for analytic potential systems, J. Differential Equations 245 (12) (2008) 3664–3673.

[14] F. Mañosas, J. Villadelprat, The bifurcation set of the period function of the dehomogenized Loud's centers is bounded, Proc. Amer. Math. Soc. 136 (2008) 1631–1642.

[15] P. Mardešić, Chebyshev Systems and the Versal Unfolding of the Cusp of Order n, Travaux en Cours, vol. 57, Hermann, 1998.

[16] P. Mardešić, D. Marín, J. Villadelprat, The period function of reversible quadratic centers, J. Differential Equations 224 (2006) 120–171.

[17] Z. Opial, Sur les périodes des solutions de l'équation différentialle $x'' + g(x) = 0$, Ann. Polon. Math. 10 (1961) 49–72.

[18] F. Rothe, Remarks on periods of planar Hamiltonian systems, SIAM J. Math. Anal. 24 (1993) 129–154.

[19] M. Sabatini, Period function's convexity for Hamiltonian centers with separable variables, Ann. Polon. Math. 85 (2005) 153–163.

[20] R. Schaaf, A class of Hamiltonian systems with increasing periods, J. Reine Angew. Math. 363 (1985) 96–109.

[21] J. Villadelprat, On the reversible quadratic centers with monotonic period function, Proc. Amer. Math. Soc. 135 (2007) 2555–2565.

[22] X. Zeng, Z. Jing, Monotonicity and critical points of period, Progr. Natur. Sci. 6 (1996) 401–407.

[23] A. Zevin, M. Pinsky, Monotonicity criteria for an energy-period function in planar Hamiltonian systems, Nonlinearity 14 (2001) 1425–1432.

[24] Y. Zhao, The period function for quadratic integrable systems with cubic orbits, J. Math. Anal. Appl. 301 (2005) 295–312.

[25] Y. Zhao, On the monotonicity of the period function of a quadratic system, Discrete Contin. Dyn. Syst. 13 (2005) 795–810.